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Invariant Forms and Hamiltonian Systems: A Geometrical Setting

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A geometric proof is given of Lee Hwa Chung's theorem for regular Hamiltonian systems, which identifies all the possible differential forms left invariant by the dynamics. Applications of this theorem in the area of canonical transformations are also remarked in a purely geometrical context.

1. INTRODUCTION

In recent years, the number of works devoted to applying the methods of differential geometry to the formulation and resolution of different kinds of physical problems and theories has been increasing. This application has been especially productive in the area of mechanics and, in particular, in the study of the dynamics of Hamiltonian systems (Abraham and Marsden, 1978; Giachetti, 1981; Godbillon, 1969; Weinstein, 1977).

Although many of the questions related to mechanics have been reformulated in this way, some aspects have not received an intrinsic treatment. In this paper, we consider one of them, an important theorem (Lee Hwa Chung, 1947) in which the integral forms left invariant by the dynamics of regular Hamiltonian systems are studied. It is useful in order to characterize the canonical transformations of these systems (Lee Hwa Chung, 1947; Gantmacher, 1975). In the original version of the theorem, its proof as well as its applications were analyzed in a local-coordinate language. We present an original geometrical study of this theme.

The paper is structured as follows: In Section 2 we compile some concepts and fundamental properties of the Hamiltonian systems. In

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Section 3 the Lee Hwa Chung theorem is stated in order to answer a question coming from a natural prosecution of several arguments in the previous section and we get a geometrical proof of it. Finally, Section 4 is devoted to showing the application of the theorem to the theory of canonical transformations in a geometrical language which reveals the deep relation between the dynamical properties and the geometry of these systems.

In our analysis only time-independent regular Hamiltonian systems with a finite number of degrees of freedom will be considered. The results that we present here have been generalized to the case of nonregular Hamiltonian systems (Gomis *et al.*, 1984; Cariñena *et al.*, 1985).

2. HAMILTONIAN SYSTEMS

The geometrical description of the Hamiltonian formalism of mechanics is realized by taking a symplectic manifold (M, Ω) as the phase space of the physical system. If such a system admits a differentiable manifold Q as configuration space, the phase space is its cotangent bundle T^*Q, which is endowed with a symplectic structure in a natural way and, in addition, this symplectic form is exact [that is, $\theta \in \Lambda^1(T^*Q)/\Omega = d\theta$, where we denote by $\Lambda^p(M)$ the set of differential p-forms in M].

Given a symplectic manifold (M, Ω), since Ω is a nondegenerate form, a canonical isomorphism $\hat{\Omega}$ is induced between the set of vector fields in M, $\mathscr{X}(M)$, and the one of 1-forms:

$$\hat{\Omega}: \mathscr{X}(M) \to \Lambda^1(M)/\hat{\Omega}(X) = \sigma \equiv i(X)\Omega, \quad \forall X \in \mathscr{X}(M)$$

[$i(X)\Omega$ means the contraction of X and Ω]. As a consequence, every function $f \in \Lambda^0(M)$ has an associated vector field $X = \hat{\Omega}^{-1}(df)$, but the converse is not true. Then:

Definition 2.1. Let (M, Ω) be a symplectic manifold. $H \in \mathscr{X}(M)$ is a local Hamiltonian vector field (lHvf) iff $i(H)\Omega$ is a closed form [we will denote these forms by $Z^p(M)$]. Then the lemma of Poincaré ensures that, for every point $m \in M$, there are a neighborhood $U \subset M$ and a function $f \in \Lambda^0(M)$ such that $i(H)\Omega|_U = df|_U$. Such a function is called a local Hamiltonian function (lHf) and the terna (M, Ω, H) is a local Hamiltonian system (lHs).

If $i(H)\Omega$ is an exact form, then H is a global Hamiltonian vector field and there is a global Hamiltonian function $f \in \Lambda^0(M)$ such that $i(H)\Omega = df$.

We will denote by $\mathscr{X}_{\rm lH}(M)$ and $\mathscr{X}_{\rm H}(M)$ the sets of local and global Hamiltonian fields, respectively, and it is evident that $\mathscr{X}_{\rm H}(M) \subset \mathscr{X}_{\rm lH}(M)$.

It is suitable to remember (Abraham and Marsden, 1978; Godbillon, 1969) that, given a differentiable manifold M, a vector field $X \in \mathscr{X}(M)$, and a p-form $\alpha \in \Lambda^p(M)$, we say that α is an absolute invariant form by X iff $L(X)\alpha = 0$ (Lie derivative). This is equivalent to $F_t^*\alpha = 0$, where $\{F_t\}$ denotes the uniparametric group of diffeomorphisms generated by the flux of X. In a similar way, we say that α is a relative invariant form by X iff $d\alpha$ is an absolute invariant form by X.

At this point we can state the following.

Theorem 2.2. Let (M, Ω) be a symplectic manifold and $X \in \mathscr{X}(M)$. Then the necessary and sufficient condition (nsc) for $X \in \mathscr{X}_{\mathrm{lH}}(M)$ is $L(X)\Omega = 0$.

Proof. Immediate because Ω is a closed form and hence

$$L(X)\Omega = i(X)d\Omega + di(X)\Omega = di(X)\Omega = 0 \Leftrightarrow i(X)\Omega \in Z^1(M)$$

$$\Leftrightarrow X \in \mathscr{X}_{\mathrm{lH}}(M) \quad \blacksquare$$

This is a very important result because it relates the locally Hamiltonian character of vector fields to the closeness of the symplectic form. An obvious corollary is:

Corollary 2.3. Let (M, Ω) be an exact symplectic manifold with $\Omega = d\theta$ and $X \in \mathscr{X}(M)$. Then $X \in \mathscr{X}_{\mathrm{lH}}(M)$ if and only if θ is a relative invariant form by X.

3. LEE HWA CHUNG THEOREM

We have just related the lHvf to the invariance of the symplectic form. Now we can search for other possible forms invariant by the set of lHvf. The answer to this question is given by a theorem which was proved (in a local-coordinate way) by Lee Hwa Chung. We prove it geometrically, in a coherent way with the rest of the formalism.

Theorem 3.1 (Lee Hwa Chung). Hypothesis: Let (M, Ω) be a connected symplectic manifold and $\alpha \in \Lambda^p(M)$ be a nondegenerate form such that it is an absolute invariant form by every $X \in \mathscr{X}_{\mathrm{lH}}(M)$.

Thesis: If $p = 2r$ [$r \in \mathbb{N} - (0)$; that is, p is even], then $\alpha = c(\wedge\Omega)^r$ ($c \equiv$ ctn).

If $p = 2r - 1$ [$r \in \mathbb{N} - (0)$; that is, p is odd], then $\alpha = 0$.

In order to prove this theorem, we need the following results:

Lemma 3.2. With the hypothesis of the theorem:

(i) For every lHvf $X_h \in \mathscr{X}_{\mathrm{lH}}(M)$ with lHf $h \in \Lambda^0(M)$ (in any neighborhood $U \subset M$), there exists a unique associated form defined from α as $\sigma_h \equiv i(X_h)\alpha$, and whose local expression (in $U \subset M$) is $\sigma_h = dh \wedge \beta$, where β is a $(p-2)$-form which is independent of X_h (it just depends on α).

(ii) In turn, the form $\beta \in \Lambda^{p-2}(M)$ is an absolute invariant form by the set $\mathscr{X}_{\mathrm{lH}}(M)$.

Proof. (i) Since α is nondegenerate, it induces an isomorphism $\hat{\alpha}: \mathscr{X}(M) \to \Lambda^{p-1}(M)$ such that, $\forall X \in \mathscr{X}(M)$, there is a unique $(p-1)$-form given by

$$\sigma = \alpha(X) \equiv i(X)\alpha$$

Consider now, $\forall X \in \mathscr{X}_{\mathrm{lH}}(M)$, the corresponding associated form $\sigma = i(X)\alpha$. Then, since α is invariant by $\mathscr{X}_{\mathrm{lH}}(M)$, we have

$$0 = L(X)\alpha = di(X)\alpha + i(X)\, d\alpha = d\sigma + i(X)\, d\alpha \Leftrightarrow d\sigma = -i(X)\, d\alpha \quad (1)$$

Now, taking two arbitrary functions $f, g \in \Lambda^0(M)$ and their product $fg \equiv h \in \Lambda^0(M)$, we have

$$dh = d(fg) = f\,dg + g\,df$$

and the Hvf associated to h must be

$$i(X_h)\Omega = dh = f\,dg + g\,df = fi(X_g)\Omega + gi(X_f)\Omega$$
$$= i(fX_g)\Omega + i(gX_f)\Omega = i(fX_g + gX_f)\Omega \Leftrightarrow X_h = fX_g + gX_f$$

Now, contracting this vector field and α, we obtain the $(p-1)$-form associated to X_h, which can be expressed as a function of the corresponding ones to X_f and X_g in the following way:

$$\sigma_h = i(X_h)\alpha = i(fX_g + gX_f)\alpha = fi(X_g)\alpha + gi(X_f)\alpha = f\sigma_h + g\sigma_f$$

and then

$$d\sigma_h = df \wedge \sigma_g + f \wedge d\sigma_g + dg \wedge \sigma_f + g \wedge d\sigma_f$$

But, on the other hand, $X_h \in \mathscr{X}_{\mathrm{lH}}(M)$ and according to the hypothesis $L(X_h)\alpha = 0$; therefore, taking into account (1),

$$d\sigma_h = -i(X_h)\, d\alpha = -i(fX_g + gX_f)\, d\alpha$$
$$= f(-i(X_g)\, d\alpha) + g(-i(X_f)\, d\alpha) = f\, d\sigma_g + g\, d\sigma_f$$

Comparing the two last expressions, we conclude

$$df \wedge \sigma_g + dg \wedge \sigma_f = 0 \quad (2)$$

Since f, g are arbitrary functions, we can take $f = g$ and then the last expression reduces to

$$2df \wedge \sigma_f = 0 \Leftrightarrow \sigma_f = df \wedge \beta_f, \qquad \forall f \in \Lambda^0(M)$$

where $\beta_f \in \Lambda^{p-2}(M)$. But this result holds $\forall f$; then, coming back to (2), we obtain

$$df \wedge dg \wedge \beta_g + dg \wedge df \wedge \beta_f = df \wedge dg \wedge (\beta_g - \beta_f) = 0,$$
$$\forall f, g \in \Lambda^0(M) \Leftrightarrow \beta_g - \beta_f = 0$$

and we conclude that β does not depend on the Hamiltonian functions or, what means the same thing, on the Hamiltonian vector fields.

(ii) Let X_f, X_g be two arbitrary lHvf, $f, g \in \Lambda^0(M)$ their lH functions (on any neighborhood $U \subset M$) and $\sigma_f = i(X_f)\alpha$, $\sigma_g = i(X_g)\alpha$ the associated $(p-1)$-forms which, as we have just seen, can be expressed as $\sigma_f = df \wedge \beta$ and $\sigma_g = dg \wedge \beta$. Then

$$L(X_g)\sigma_f = L(X_g)(df \wedge \beta) = L(X_g)\, df \wedge \beta - df \wedge L(X_g)\beta$$

but taking into account that $L(X_g)\, df = dL(X_g)f$ and that $L(X_g)f = \{f, g\}$ we obtain

$$L(X_g)\sigma_f = d\{f, g\} \wedge \beta - df \wedge L(X_g)\beta \tag{3}$$

On the other hand, by using the definition of σ_f, we obtain

$$L(X_g)\sigma_f = L(X_g)i(X_f)\alpha = i([X_g, X_f])\alpha + i(X_f)L(X_g)\alpha = i([X_g, X_f])\alpha$$

where the fact that $L(X_g)\alpha = 0$ as well as a known property of the differential operators is used. But also $[X_g, X_f] = X_{\{f,g\}}$; thereby

$$L(X_g)\sigma_f = i([X_g, X_f])\alpha = i(X_{\{f,g\}})\alpha = \sigma_{\{f,g\}} = d\{f, g\} \wedge \beta$$

and comparing with (3), we can conclude

$$df \wedge L(X_g)\beta = 0; \qquad \forall f \in \Lambda^0(M), \qquad \forall X_g \in \mathscr{X}_{\mathrm{lH}}(M)$$

whence $L(X_g)\beta = 0$, $\forall X_g \in \mathscr{X}_{\mathrm{lH}}(M)$. ■

Using these results, the proof of the theorem is performed as follows:

Proof of Lee Hwa Chung theorem. We distinguish two cases:

(a) If $p = 2r$. Following an induction procedure, we first prove:

If $r = 1$, *then* $\alpha = C_1\Omega$ $(C_1 = ctn)$

In fact, if $H \in \mathscr{X}_{\mathrm{lH}}(M)$, we have its associated form $\sigma_h = dh \wedge \beta \in \Lambda^1(M)$ and then $\beta \in \Lambda^0(M)$, which, according to Lemma 3.2, is invariant by $\mathscr{X}_{\mathrm{lH}}(M)$; hence,

$$0 = L(X)\beta = di(X)\beta + i(X)\, d\beta = i(X)\, d\beta, \qquad \forall X \in \mathscr{X}_{\mathrm{lH}}(M)$$

but, taking into account that we can take a local basis of $\mathscr{X}(M)$ made up by lHvf, this result is also valid $\forall X \in \mathscr{X}(M)$, and thus we have to conclude that $d\beta = 0$ and $\beta = C_1$ (ctn). Hence,

$$\sigma_h = C_1\, dh = C_1 i(H)\Omega = i(H)C_1\Omega$$

On the other hand, $\sigma_{\rm H} = i(H)\alpha$; therefore, comparing both results, we conclude that

$$i(H)\alpha = i(H)C_1\Omega, \qquad \forall H \in \mathscr{X}_{\rm lH}(M) \Leftrightarrow \alpha = C_1\Omega$$

If we assume now that the theorem holds for $r-1$, we next prove that it holds for r; that is:

If every form of degree $2(r-1)$ being absolute invariant by $\mathscr{X}_{\rm lH}(M)$ can be expressed as $\alpha^{2(r-1)} = C_{r-1}(\wedge\Omega)^{r-1}$ ($C_{r-1} = \text{ctn}$), then every form of degree $2r$ being absolute invariant by $\mathscr{X}_{\rm lH}(M)$ can be expressed as $\alpha^{2r} = C_r(\wedge\Omega)^r$ ($C_r = \text{ctn}$).

In fact, let us consider the form $\alpha^{2r} \in \Lambda^{2r}(M)$ and the corresponding one associated to an arbitrary vector field $H \in \mathscr{X}_{\rm lH}(M)$. Once again from Lemma 3.2 we have that $\sigma_{\rm H} = dh \wedge \beta$ is now $\beta \in \Lambda^{2(r-1)}(M)$, which is invariant by $\mathscr{X}_{\rm lH}(M)$. Hence, by the hypothesis, $\beta = C_{r-1}(\wedge\Omega)^{r-1}$ and, consequently,

$$\begin{aligned}\sigma_h &= C_{r-1}\, dh(\wedge\Omega)^{r-1} = C_{r-1}(i(H)\Omega)(\wedge\Omega)^{r-1}\\ &= (1/r)C_{r-1}i(H)(\wedge\Omega)^r\\ &= i(H)(C_r(\wedge\Omega)^r)\end{aligned}$$

where use is made of the property

$$i(H)(\wedge\Omega)^r = r(i(H)\Omega)(\wedge\Omega)^{r-1}$$

and we have put $C_r \equiv (1/r)C_{r-1}$. Now, since $\sigma_h = i(H)\alpha^{2r}$, comparing the last expressions, we have

$$i(H)(C_r(\wedge\Omega)^r) = i(H)\alpha^{2r} \qquad \forall H \in \mathscr{X}_{\rm lH}(M) \Leftrightarrow \alpha^{2r} = C_r(\wedge\Omega)^r$$

(b) If $p = 2r-1$. Following a similar procedure, we first have:

If $r = 1$, then $\alpha = 0$.

In fact, the form $\sigma_h = i(H)\alpha$, $\forall H \in \mathscr{X}_{\rm lH}(M)$, is now a function; then from (2) we obtain

$$0 = dh \wedge \sigma_h, \quad \forall h \in \Lambda^0(M), \Leftrightarrow \sigma_h = 0$$

(since the alternative $\sigma_h = dh \wedge \beta$ is impossible because σ_h is a function). Then we have

$$0 = \sigma_h = i(H)\alpha, \quad \forall H \in \mathscr{X}_{\rm lH}(M), \Leftrightarrow \alpha = 0$$

Assuming that the result holds for $r-1$, we are going to prove it holds for r, that is:

If every form of degree $2(r-1)-1$ that is absolute invariant by $\mathscr{X}_{\rm lH}(M)$ is null identically, then every form of degree $2r-1$ that is absolute invariant by $\mathscr{X}_{\rm lH}(M)$ is also null identically.

In fact, let us consider α^{2r-1} being invariant by $\mathscr{X}_{\mathrm{lH}}(M)$ and the associated form $\sigma_h = i(H)\alpha^{2r-1} \in \Lambda^{2r-2}(M)$, $H \in \mathscr{X}_{\mathrm{lH}}(M)$. We have $\sigma_h = dh \wedge \beta$ with $\beta \in \Lambda^{2(r-1)-1}(M)$, which is invariant by $\mathscr{X}_{\mathrm{lH}}(M)$; therefore, by the hypothesis, $\beta = 0$, and, necessarily, $\sigma_h = 0$. Then

$$\sigma_h = i(H)\alpha^{2r-1} = 0, \quad \forall H \in \mathscr{X}_{\mathrm{lH}}(M) \Leftrightarrow \alpha^{2r-1} = 0$$

and the proof is finished. ■

Hence, the theorem identifies all the absolute invariant forms by every lHvf (their degree is even). Now we can demand all the relative invariant ones. The following immediate corollary of the previous theorem gives the answer to this problem:

Corollary 3.3. Hypothesis: Let (M, Ω) be a connected exact symplectic manifold with $\Omega = d\theta$ and $\alpha \in \Lambda^p(M)$ a nondegenerate form such that it is a relative invariant form by every $X \in \mathscr{X}_{\mathrm{lH}}(M)$.

Thesis: If $p = 2r-1$ [$r \in \mathbb{N} - (0)$; that is, p is odd], then $\alpha = C\theta(\wedge\Omega)^{r-1}$ ($C = \text{ctn}$).

If $p = 2r$ [$r \in \mathbb{N} - (0)$; that is, p is even], then $\alpha = 0$.

Thus, all the relative invariant forms by $\mathscr{X}_{\mathrm{lH}}(M)$ are odd degree forms.

4. CANONICAL AND SYMPLECTIC TRANSFORMATIONS

We have seen in Section 2 how the dynamics of a physical system (which is determined by the lHvf) is related to the symplectic structure of the phase space of this system. Next we want to study the relation between the more interesting transformations concerning the dynamics of the systems and those concerning the geometry of their corresponding phase spaces.

Attending to the dynamics we define:

Definition 4.1. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds. A map $\Phi \in C^\infty(M_1, M_2)$ is a canonical transformation (ct) from M_1 to M_2 iff (i) Φ is a diffeomorphism (and hence $\dim M_1 = \dim M_2$), and (ii) the differential application $\Phi_*: \mathscr{X}(M_1) \to \mathscr{X}(M_2)$ transforms (biunivocally) the lHvf of (M_1, Ω_1) into the lHvf of (M_2, Ω_2), that is, $\Phi_*(\mathscr{X}_{\mathrm{lH}}(M_1)) = \mathscr{X}_{\mathrm{lH}}(M_2)$.

From the geometrical point of view, the more interesting transformations are the following ones:

Definition 4.2. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds. A map $\Phi \in C^\infty(M_1, M_2)$ is a symplectic transformation from M_1 to M_2 iff (i) Φ is a diffeomorphism (hence $\dim M_1 = \dim M_2$ and we also call Φ a symplectomorphism) and (ii) $\Phi^*\Omega_2 = \Omega_1$.

Although the origin of these concepts is different, the relation between the lHvf and the geometry (Theorem 2.2) induces one to think that a relation between them must exist, too. The Lee Hwa Chung theorem allows us to find such a relation.

Theorem 4.3. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds and $\Phi \in C^\infty(M_1, M_2)$ a diffeomorphism. The nsc for Φ to be a ct is that $\Phi^*\Omega_2 = C\Omega_1$ $(C = \text{ctn})$.

Proof. $(\Rightarrow)$: If Φ is a ct, it implies $\Phi_* H_1 = H_2 \in \mathscr{X}_{\text{lH}}(M_2)$, $\forall H_1 \in \mathscr{X}_{\text{lH}}(M_1)$, and according to Theorem 2.2, we have $L(H_2)\Omega_2 = 0$; then

$$0 = \Phi^* L(H_2)\Omega_2 = L(\Phi_*^{-1} H_2)\Phi^*\Omega_2 = L(H_1)\Phi^*\Omega_2$$

and from the Lee Hwa Chung theorem we conclude that $\Phi^*\Omega_2 = C\Omega_1$.

$(\Leftarrow)$: Conversely, $\forall H_1 \in \mathscr{X}_{\text{lH}}(M_1)$ we have that $L(H_1)\Omega_1 = 0$ and, since $\Phi^*\Omega_2 = C\Omega_1$ (by the hypothesis), we have

$$0 = \Phi^* L(H_1)\Omega_1 = L(\Phi_* H_1)\Phi^{*-1}\Omega_1 = L(\Phi_* H_1)\Omega_2 \Leftrightarrow L(\Phi_* H_1)\Omega_2 = 0$$

Therefore, $\Phi_* H_1 \in \mathscr{X}_{\text{lH}}(M_2)$ and hence Φ is a ct. ■

The constant C is the valence of the ct and when $C = 1$ we have a univalent or restricted canonical transformation (Saletan and Cromer, 1971).

The equivalence between the aforementioned concepts is an immediate consequence of this result:

Theorem 4.4. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds and $\Phi \in C^\infty(M_1, M_2)$ a diffeomorphism. The following statements are equivalent: (a) Φ is a univalent canonical transformation, (b) Φ is a symplectomorphism.

Theorem 4.3 leads to other interesting consequences that are sometimes used as tests in order to inquire into the canonical character of a transformation. The first concerns the Poisson brackets between Hamiltonian functions (which, in a symplectic manifold, are all the functions because of the canonical isomorphism). Previously we took into account the following fact:

Proposition 4.5. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds and $\Phi \in C^\infty(M_1, M_2)$ a ct and $H_1 \in \mathscr{X}_{\text{lH}}(M_1)$. If $h_1 \in \Lambda^0(M_1)$ is a lHf of H_1 in a neighborhood $U_1 \subset M_1$, then every lHf $h_2 \in \Lambda^0(M_2)$ of $H_2 = \Phi_* H_1 \in \mathscr{X}_{\text{lH}}(M_2)$ in the neighborhood $U_2 = \Phi(U_1) \subset M_2$ is related to h_1 by $Ch_1 = \Phi^* h_2 + k$ $(C, k$ ctn).

Proof. By hypothesis, $i(H_1)\Omega_1|_{U1} = dh_1|_{U1}$; then, taking into account Theorem 4.3, we have

$$\Phi^{*-1}(i(H_1)\Omega_1)|_{U1} = i(\Phi_* H_1)(\Phi^{*-1}\Omega_1)|_{\Phi(U1)} = i(H_2)\Omega_2|_{U2}$$

On the other hand, $\Phi^{*-1}\, dh_1 = d(\Phi^{*-1} h_1)$, and hence

$$i(H_2)\Omega_2|_{U2} = Cd(\Phi^{*-1} h_1)|_{U2} = d(\Phi^{*-1} Ch_1)|_{U2}$$

but $H_2 \in \mathscr{X}_{\mathrm{IH}}(M_2)$, therefore $\exists h_2 \in \Lambda^0(M_2)$ such that $i(H_2)\Omega_2|_{U2} = dh_2|_{U2}$, and then the result follows immediately. ■

Now the main result is:

Theorem 4.6. Let (M_1, Ω_1) and (M_2, Ω_2) be symplectic manifolds and $\Phi \in C^\infty(M_1, M_2)$ a diffeomorphism. The nsc for Φ to be a ct is that

$$\Phi^*\{f_2, g_2\} = C^{-1}\{\Phi^* f_2, \Phi^* g_2\}, \qquad \forall f_2, g_2 \in \Lambda^0(M_2) \tag{4}$$

Proof. According to the properties of the Poisson bracket, we have $\forall f_2, g_2 \in \Lambda^0(M_2)$

$$\Phi^*\{f_2, g_2\} = \Phi^* L(X_{g2}) f_2 = L(\Phi_*^{-1} X_{g2})\Phi^* f_2 \tag{5}$$

($\Rightarrow$) If Φ is a ct, since $X_{g2} \in \mathscr{X}_{\mathrm{IH}}(M_2)$, then $\Phi_*^{-1} X_{g2} \in \mathscr{X}_{\mathrm{IH}}(M_2)$ and its Hamiltonian function is $C^{-1}\Phi^* g_2$ (according to Theorem 4.4); hence

$$\Phi^*\{f_2, g_2\} = L(\Phi_*^{-1} X_{g2})\Phi^* f_2 = L(C^{-1} X_{\Phi^* g_2})\Phi^* f_2 = C^{-1}\{\Phi^* f_2, \Phi^* g_2\}$$

($\Leftarrow$) Conversely, if (4) holds, taking (5) into account, it is equivalent to

$$L(\Phi_*^{-1} X_{g2})\Phi^* f_2 = C^{-1} L(X_{\Phi^* g_2})\Phi^* f_2; \qquad \forall f_2, g_2 \in \Lambda^0(M_2)$$

and necessarily

$$\Phi_*^{-1} X_{g2} = C^{-1} X_{\Phi^* g_2} \in \mathscr{X}_{\mathrm{IH}}(M_1), \qquad \forall X_{g2} \in \mathscr{X}_{\mathrm{IH}}(M_2)$$

and then Φ is a ct. ■

Another property to be used as test of canonicity is given by:

Theorem 4.7. Let (M_1, Ω_1) and (M_2, Ω_2) be exact symplectic manifolds with $\Omega_1 = d\theta_1$, $\Omega_2 = d\theta_2$, and let $\Phi \in C^\infty(M_1, M_2)$ be a diffeomorphism. The nsc for Φ to be a ct is that some function $F_1 \in \Lambda^0(M_1)$ locally exists [or also $F_2 \in \Lambda^0(M_2)$] such that

$$\Phi^*\theta_2 - C\theta_1 - dF_1 = 0 \qquad (\text{or } \Phi^{*-1}\theta_1 - \theta_2 - dF_2 = 0) \tag{6}$$

F_1 and F_2 are called Poincaré generating functions of the ct and $F_1 = C\Phi^* F_2 - k$ (C, k ctn).

Proof. According to Theorem 4.3, Φ is a ct if and only if

$$0 = \Phi^*\Omega_2 - C\Omega_1 = \Phi^*\, d\theta_2 - Cd\theta_1 = d(\Phi^*\theta_2 - C\theta_1)$$

and, by the lemma of Poincaré, there exists locally $F_1 \in \Lambda^0(M_1)$ such that (6) holds. The existence of F_2 is proved in a similar way using Φ^{*-1}. Finally, comparing suitably equalities, we obtain the relation between both functions. ■

These generating functions are not the known *mixed generating functions* which appear in the classical texts (Gantmacher, 1975; Goldstein, 1950). There is a more general concept of generating function, the Weinstein generating function, which includes all these (Abraham and Marsden, 1978).

The last result relates the dynamical evolution of a Hamiltonian system to the canonical transformations:

Proposition 4.8. Let (M, Ω) be a symplectic manifold. A vector field $H \in \mathscr{X}(M)$ is a lHvf if and only if its flux is a group of infinitesimal univalent canonical transformations.

Proof. Immediate because $L(H)\Omega = 0 \Leftrightarrow F_t^*\Omega - \Omega = 0$, $\forall H \in \mathscr{X}(M)$, which, according to Theorems 2.2 and 4.3, proves the assertion. ■

5. CONCLUSIONS

We have presented an original intrinsic proof of the Lee Hwa Chung theorem and we have shown that this theorem plays a role in identifying the canonical transformations (the more interesting ones for the dynamics) and the symplectic transformations (the more interesting ones for the geometry). This (known) result has been presented intrinsically. Subsequently, we have shown, in this geometrical context, how to obtain the several tests of canonicity, as well as other properties of the ct, as applications of the aforementioned theorem.

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REFERENCES

Abraham, R., and Marsden, J. E. (1978). *Foundation of Mechanics*, Benjamin-Cummings, Reading, Massachusetts.

Cariñena, J. F., Gomis, J., Ibort, L. A., and Roman, N. (1985). *Journal of Mathematical Physics*, **26**, 1961.

Gantmacher, F. (1975). *Lectures in Analytical Mechanics*, Mir, Moscow.

Giachetti, R. (1981). *Nuovo Cimento*, **4**, 12 (1981).

Godbillon, C. (1969). *Géometrie Differentielle et Mécanique Analytique*, Hermann, Paris.

Goldstein, H. (1950). *Classical Mechanics*, Addison-Wesley, Reading, Massachusetts.
Gomis, J., Llosa, J., and Roman, N. (1984). *Journal of Mathematical Physics* **25**, 1348.
Lee Hwa Chung (1947). *Proceedings of the Royal Society*, **LXIIA**, 237.
Saletan, E. J., and Cromer, A. H. (1971). *Theoretical Mechanics*, Wiley, New York.
Weinstein, A. (1977). Lectures on symplectic manifolds, *CBMS Regional Conference Series Mathematics*, **29**, 237.